CHAPTER 10

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Chapter Objectives

- Introduction
- Picard's method
- Taylor's series method
- Euler's method
- Modified Euler's method
- Runge's method
- Runge-Kutta method
- Predictor-corrector methods.
- Milne's method
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- Simultaneous first order differential equations
- Second order differential equations.
- Error analysis
- Convergence of a method
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- Boundary-value problems
- Finite-difference method
- Shooting method
- Objective type of questions

10.1 Introduction

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realize that computing machines are now readily available which reduce numerical work considerably.

Solution of a differential equation. The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the *closed* or *finite form of solution*. In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$$dy/dx = f(x, y), \text{ given } y(x_0) = y_0 \tag{1}$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions *either* as a power series in x from which the values of y can be found by direct substitution, *or* a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x. *The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as* **single-step methods.**

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth, etc. belong to the latter class of solutions. *In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations until sufficient accuracy is achieved. As such, these methods are called* **step-by-step methods**.

Euler and Runga-Kutta methods are used for computing y over a limited range of x- values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values. Therefore Milne and Adams methods require starting values which are found by Picard's Taylor series or Runge-Kutta methods.

Initial and boundary conditions. An ordinary differential equation of the nth order is of the form

$$F(x, y, dy/dx, d^2y/dx^2, \cdots, d^n y/dx^n) = 0 \tag{2}$$

Its general solution contains n arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \cdots, c_n) = 0 \tag{3}$$

To obtain its particular solution, n conditions must be given so that the constants c_1, c_2 …, c_n can be determined.

If these conditions are prescribed at one point only (say:x_0), then the differential equation together with the conditions constitute an **initial value problem** *of the nth order.*

If the conditions are prescribed at two or more points, then the problem is termed as **boundary value problem.**

In this chapter, we shall first describe methods for solving initial value problems and then explain the **finite difference method** and **shooting method** for solving boundary value problems.

10.2 Picard's Method

Consider the first order equation $\dfrac{dy}{dx} = f(x, y)$ (1)

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int dy \aleph \int f(x,y)dx \text{ or } y \quad y_0 \quad \int f(x,y)dx \tag{2}$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y = y + \int_{x_0}^{x} f\ x\ y\ dx$$

Similarly, *a third approximation is*

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

Continuing this process, we obtain y_4, y_5, $\cdots$ y_n where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

Hence this method gives a sequence of approximations y_1, y_2, y_3 $\cdots$ each giving a better result than the preceding one.

NOTE **Obs.** *Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See Sections 10.11 and 10.12).*

EXAMPLE 10.1

Using Picard's process of successive approximations, obtain a solution up to the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.

Solution:

(i) We have $y = 1 + \int_{x_0}^{x} (x + y)dx$

First approximation. Put $y = 1$ in $y + x$, giving

$$y_1 = 1 + \int_{x_0}^{x} (1 + x)dx = 1 + x + x^2/2$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_1 = 1 + \int_{x_0}^{x} (1 + x + x^2/2)dx = 1 + x + x^2 + x^3/6$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_{x_0}^{x} (1 + x + x^2 + x^3/6)dx = 1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_{0}^{x} \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right)dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

Fifth approximation, Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx \quad (1)$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

(*ii*) Given equation

$$\frac{dy}{dx} - y = x \text{ is a Leibnitzs linear in } x$$

Its, I.F. being e^{-x} the solution is

$$ye^{-x} = \int xe^{-x} dx + c$$

$$= -xe^{-x} - \int \left(-e^{-x}\right) dx + c = -xe^{-x} - e^{-x} + c$$

$\therefore \qquad y = ce^x - x - 1$

Since $y = 1$, when $x = 0$, $\qquad \therefore c = 2$.

Thus the desired particular solution is

$$y = 2e^x - x - 1 \quad (2)$$

Or using the series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

We get $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \cdots\infty \quad (3)$$

Comparing (1) and (3), it is clear that (1), approximates to the exact particular solution (3) upto the term in x^5.

NOTE **Obs.** *At* $x = 1$, *the fourth approximation* $y_4 = 3.433$ *and the fifth approximation* $y_5 = 3.434$ *whereas the exact value is* 3.44.

EXAMPLE 10.2

Find the value of y for $x = 0.1$ by Picard's method, given that

$$\frac{dy}{dx} = \frac{y - x}{y + x}, y(0) = 1.$$

Solution:

We have $y = 1 + \int_0^x \frac{y - x}{y + x} dx$

First approximation. Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{y-x}{y+x}dx = 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right)dx$$

$$= 1 + \left[-x + 2\log(1+x)\right]_0^x = 1 - x + 2\log(1+x)$$

Second approximation. Put $y = 1 - x + 2\log(1+x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2\log(1+x) - x}{1 - x + 2\log(1+x) + x}dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1+x)}\right]dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (i) we obtain

$$y(0.1) = 1 - (0.1) + 2\log 1.1 = 0.9828.$$

10.3 Taylor's Series Method

Consider the first order equation $\dfrac{dy}{dx} = f(x,y)$ (1)

Differentiating (1), we have $\dfrac{d^2y}{dx^2} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$ *i.e.* $y'' = f_x + f_y f'$ (2)

Differentiating this successively, we can get y'', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the

Values of $(y')_0, (y'')_0, (y''')_0$ can be obtained. Hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots \quad (3)$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_i$ from (3), y', y'' etc. can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

NOTE

Obs. *This is a single step method and works well so long as the successive derivatives can be calculated easily. If (x, y) is somewhat complicated and the calculation of higher order derivatives becomes tedious, then Taylor's method cannot be used gainfully. This is the main drawback of this method and therefore, has little application for computer programs. However, it is useful for finding starting values for the application of powerful methods like Runga-Kutta, Milne and Adams- Bashforth which will be described in the subsequent sections.*

EXAMPLE 10.3

Solve $y' = x + y$, $y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution:

Differentiating successively, we get

$$y' = x + y \qquad y'(0) = 1 \qquad [\because y(0) = 1]$$
$$y'' = 1 + y' \qquad y''(0) = 2$$
$$y''' = y'' \qquad y'''(0) = 2$$
$$y''' = y''' \qquad y'''(0) = 2, \text{ etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \cdots$$

Here $x_0 = 0$, $y_0 = 1$

$$\therefore \qquad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(4)\cdots$$

Thus
$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}\cdots$$
$$= 1.1103$$

and
$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \cdots$$
$$= 1.2427$$

EXAMPLE 10.4

Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $dy/dx = x^2y - 1$, $y(0) = 1$.

Solution:

Differentiating successively, we get

$$y' = x^2y - 1, \qquad (y')_0 = -1 \qquad [\because y(0) = 1]$$

$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$

$$y''' = 2y + 4xy' + x2y'', \qquad (y''')_0 = 2$$

$$y^{iv} = 6y' + 6xy'' + x2y''', \qquad (y^{iv})_0 = -6, \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(-6) + \cdots$$

$$= 1 + -x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence $y(0.1) = 0.90033$ and $y(0.21) = 0.80227$

EXAMPLE 10.5

Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

Solution:

(a) We have $y' = 2y + 3e^x$; $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0$, $y = 0$ we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$

$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$

$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \cdots$$

$$= 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{15}{8}x^4 + \cdots$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.2)^4 + \cdots = 0.8110$ (i)

(b) Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x

Its I.F. being e_{-2x}, the solution is

$$ye^{-2x} = \int 3e^{x} \cdot e^{-2x}\, dx + c = -3e^{-x} + c \text{ or } \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\therefore c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112$ (ii)

Comparing (i) and (ii), it is clear that (i) approximates to the exact value up to three decimal places

EXAMPLE 10.6

Solve by Taylor series method of third order the equation $\frac{dy}{dx} = \frac{x^3 + xy^2}{e^x}$, $y(0) = 1$ for y at $x = 0.1$, $x = 0.2$ and $x = 0.3$

Solution:

We have $\quad y' = \left(x^3 + xy^2\right)e^{-x}; \quad y'(0) = 0$

Differentiating successively and substituting $x = 0$, $y = 1$.

$$y'' = (x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x.2y.y')e^{-x}$$
$$= (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')e^{-x}; \quad y''(0) = 1$$
$$y''' = (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')(-e^{-x})$$
$$+\{-3x^2 - (y^2 + x.2y.y') + 6x + 2yy'$$
$$+2[yy' + x(y'^2 + yy'')]\}e^{-x} \quad y'''(0) = -2$$

Substituting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$
$$= 1 + x(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-2) + \cdots$$
$$= 1 + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

Hence $y(0.1)=1+\frac{1}{2}(0.1)^2-\frac{1}{3}(0.1)^3=1.005$

$$y(0.2)=1+\frac{1}{2}(0.2)^2-\frac{1}{3}(0.2)^3=1.017$$

$$y(0.3)=1+\frac{1}{2}(0.3)^2-\frac{1}{3}(0.3)^3=1.036$$

EXAMPLE 10.7

Solve by Taylor's series method the equation $\frac{dy}{dx}=\log(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1) = 2$.

Solution:

We have $y' = \log x + \log y$; $y'(1) = \log 2$

Differentiating w.r.t., x and substituting $x = 1$, $y = 2$, we get

$$y''=\frac{1}{x}+\frac{1}{y}y' \quad y''=1+\frac{1}{2}\log 2$$

$$y'''=\frac{1}{x^2}+\frac{1}{y}y''+y'\left(-\frac{1}{y^2}\right)y';$$

$$y'''=1+\frac{1}{2}\left(1+\frac{1}{2}\log 2\right)-\frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $s = 1$, we have

$$y(x)=y(1)+(x-1)y'(1)+\frac{(x-1)^2}{2!}y''(1)+\frac{(x-1)^3}{3!}y'''(1)+\cdots$$

$$=2+(x-1)\log 2+\frac{1}{2}(x-1)^2\left(1+\frac{1}{2}\log 2\right)$$

$$+\frac{1}{6}(x-1)^3\left[-\frac{1}{2}+\frac{1}{4}\log 2-\frac{1}{4}(\log 2)^2\right]$$

$$\therefore\ y(1.1)=2+(0.1)\log 2+\frac{(0.1)^2}{2}\left(1+\frac{1}{2}\log 2\right)+\frac{(0.1)^3}{6}\left[-\frac{1}{2}+\frac{1}{4}\log 2-\frac{1}{4}(\log 2)^2\right]$$

$$=2.036$$

$$y(1.2)=2+(0.2)\log 2+\frac{(0.2)^2}{2}\left(1+\frac{1}{2}\log 2\right)+\frac{(0.2)^3}{6}\left[-\frac{1}{2}+\frac{1}{4}\log 2-\frac{1}{4}(\log 2)^2\right]$$

$$=2.081$$

Exercises 10.1

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0$, $y_0 = 1$ up to the third approximation.
2. Employ Picard 's method to obtain, correct to four places of decimals the, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$.
3. Obtain Picard's second approximate solution of the initial value problem
$$y' = x^2/(y^2 + 1),\ y(0)=0.$$
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using
   (*a*) Picard's method (*b*) Taylor's series.
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value.
6. Using Taylor's series method, compute $y(0.2)$ to three places of decimals from $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$.
7. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y\ (x)$ satisfies
$$y' = xy + 1,\ y(0) = 1.$$
8. Solve $y' = y^2 + x$, $y(0) = 1$ using Taylor's series method and compute $y(0.1)$ and $y(0.2)$.
9. Evaluate $y(0.1)$ correct to four decimal places using Taylor's series methods if $dy/dx = x^2 + y^2$, $y(0) = 1$.
10. Using Taylor series method, find $y(0.1)$ correct to three decimal places given that $dy/dx = e^x - y^2$, $y(0) = 1$

10.4 Euler's Method

Consider the equation $\frac{dy}{dx} =$ ˘˘ (1)

given that $y(x_0) = y_0$.Its curve of solution through $P(x_0, y_0)$is shown dotted in Figure.10.1. Now we have to find the ordinate of any other point Q on this curve.

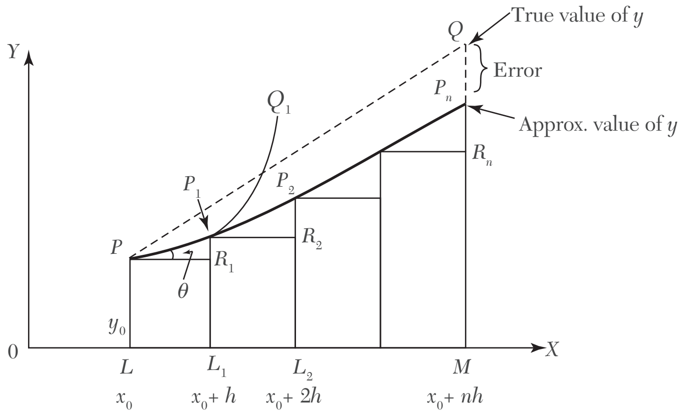

FIGURE 10.1

Let us divide LM into n sub-intervals each of width h at $L_1, L_2 \cdots$ so that h is quite small

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_p = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0 + h, y_1) \tag{1}$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

NOTE **Obs.** *In Euler's method, we approximate the curve of solution by the tangent in each interval, i.e., by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. As such, the method is very slow and hence there is a modification of this method which is given in the next section.*

EXAMPLE 10.8

Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

x	y	$x + y = dy/dx$	Old y + 0.1 (dy/dx) = *new* y
0.0	1.00	1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

NOTE

Obs. *In Example 10.1(Obs.), we obtained the true values of y from its exact solution to be 3.44 where as by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labor of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.*

EXAMPLE 10.9

Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Solution:

We divide the interval (0, 0.1) in to five steps, *i.e.*, we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows:

x	y	dy/dx	$Old y + 0.02\ (dy/dx) = new\ y$
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(0.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(0.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(0.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

10.5 Modified Euler's Method

In Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Figure 10.1) such that at P_1, we have

$$y_1 = y_0 + h\,f(x_0, y_0) \tag{1}$$

Then the slope of the curve of solution through P_1

$$[\text{i.e., } (dy/dx)P_1 = f(x_0 + h, y_1)]$$

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in

$$P_2(x_0 + 2h, y_2).$$

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)]$$

As the slope of the tangent at $P1$ is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1(1)$

Again (2) is applied and we find a still better value $y_{1(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, until two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to a desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which gives great improvement in accuracy over the original method.

EXAMPLE 10.10

Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

The various calculations are arranged as follows taking $h = 0.1$:

x	$x + y = y'$	*Mean slope*	*Old y + 0.1 (mean slope) = new y*
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}(1+1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}(1+1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}(1+1.2105)$	1.00 + 0.1 (1.1052) = 1.1105
Since the last two values are equal, we take $y(0.1) = 1.1105$.			
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}(1.12105+1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}(1.2105+1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}(1.2105+1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432
Since the last two values are equal, we take $y(0.2) = 1.2432$.			
0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}(1.4432+1.6875)$	1.2432 + 0.1 (1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}(1.4432+1.6997)$	1.2432 + 0.1 (1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}(1.4432+1.7003)$	1.2432 + 0.1 (1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}(1.4432+1.7004)$	1.2432 + 0.1 (1.5718) = 1.4004
Since the last two values are equal, we take $y(0.3) = 1.4004$.			

Hence $y(0.3) = 1.4004$ approximately.

NOTE **Obs.** *In Example 10.8, we obtained the approximate value of y for x = 0.3 to be 1.53 whereas by the modified Euler's method the corresponding value is 1.4003 which is nearer its true value 1.3997, obtained from its exact solution y = 2ex – x – 1 by putting x = 0.3.*

EXAMPLE 10.11

Using the modified Euler's method, find $y(0.2)$ and $y(0.4)$ given

$$y' = y + e^x,\ y(0) = 0.$$

Solution:

We have $y' = y + ex = f(x, y)$; $x = 0$, $y = 0$ and $h = 0.2$

The various calculations are arranged as under:

To calculate $y(0.2)$:

x	$y + ex = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.0	1	—	$0 + 0.2\ (1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + 0.2\ (1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2\ (1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2\ (1.2338) = 0.2468$
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2\ (1.2341) = 0.2468$

Since the last two values of y are equal, we take $y\ (0.2) = 0.2468$.

To calculate $y(0.4)$:

x	$y + ex$	*Mean slope*	*Oldy + 0.2 (mean slope) new y*
0.2	$0.2468 + e^{0.2} = 1.4682$	—	$0.2468 + 0.2\ (1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + 0.2\ (1.7502) = 0.5968$

x	$y + ex$	*Mean slope*	*Oldy* + 0.2 (*mean slope*) *new y*
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887)$ $= 1.7784$	0.2468 + 0.2 (1.7784) = 0.6025
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943)$ $= 1.78125$	0.2468 + 0.2 (1.78125) = 0.6030
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949)$ $= 1.7815$	0.2468 + 0.2 (1.7815) = 0.6031
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949)$ $= 1.7816$	0.2468 + 0.2 (1.7815) = 0.6031

Since the last two value of y are equal, we take $y(0.4) = 0.6031$

Hence $y(0.2) = 0.2468$ an d $y(0.4) = 0.6031$ approximately.

EXAMPLE 10.12

Solve the following by Euler's modified method:

$$\frac{dy}{dx} = \log(x + y), y(0) = 2$$

at $x = 1.2$ and 1.4 with $h = 0.2$.

Solution:

The various calculations are arranged as follows:

x	$\log(x + y) = y'$	*Mean slope*	*Old y* + 0.2 (*mean slope*) = *new y*
0.0	log (0 + 2)	—	2 + 0.2(0.301) = 2.0602
0.2	log (0.2 + 2.0602)	$\frac{1}{2}(0.310 = 0.3541)$	2 + 0.2 (0.3276) = 2.0655
0.2	log (0.2 + 2.0655)	$\frac{1}{2}(0.301 + 0.3552)$	2 + 0.2 (0.3281) = 2.0656
0.2	0.3552	—	2.0656 + 0.2 (0.3552) = 2.1366
0.4	log (0.4 + 2.1366)	$\frac{1}{2}(0.3552 + 0.4042)$	2.0656 + 0.2 (0.3797) = 2.1415
0.4	log (0.4 + 2.1415)	$\frac{1}{2}(0.3552 + 0.4051)$	2.0656 + 0.2 (0.3801) = 2.1416
0.4	0.4051	—	2.1416+ 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}(0.4051 + 0.4506)$	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}(0.4051 + 0.4514)$	2.1416 + 0.2 (0.4282) = 2.2272

x	$\log(x + y) = y'$	*Mean slope*	*Old* y + 0.2 (*mean slope*) = *new* y
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}(0.4514 + 0.4938)$	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}(0.4514 + 0.4943)$	2.2272 + 0.2 (0.4727) = 2.3217
0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}0.4943 + 0.5341)$	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}(0.4943 + 0.5346)$	2.3217 + 0.2 (0.5144) = 2.4245
1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}(0.5346 + 0.5719)$	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}(0.5346 + 0.5723)$	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}(0.5723 + 0.6074)$	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}(0.5723 + 0.6078)$	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ an d $y(1.4) = 2.6531$ approximately.

EXAMPLE 10.13

Using Euler's modified method, obtain a solution of the equation

$$dy/dx = x + \left|\sqrt{y}\right|$$

with initial conditions $y = 1$ at $x = 0$, for the range 0 £ x £ 0.6 in steps of 0.2.

Solution:

The various calculations are arranged as follows:

x	$x + \left\|\sqrt{y}\right\| = y'$	*Mean slope*	*Old* y + 0.2 (*mean slope*) = *new* y
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \left\|\sqrt{(1.2)}\right\|$ = 1.2954	$\frac{1}{2}(1 + 1.2954)$ = 1.1477	1 + 0.2 (1.1477) = 1.2295

x	$x+\left\|\sqrt{y}\right\|=y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.2	$0.2+\left\|\sqrt{(1.2295)}\right\|$ = 1.3088	$\frac{1}{2}(1+1.3088)$ = 1.1544	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2+\left\|\sqrt{(1.2309)}\right\|$ = 1.3094	$\frac{1}{2}(1+1.3094)$ = 1.1547	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4+\left\|\sqrt{(1.4927)}\right\|$ = 1.6218	$\frac{1}{2}(1.3094+1.6218)$ = 1.4654	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.4+\left\|\sqrt{(1.524)}\right\|$ = 1.6345	$\frac{1}{2}(1.3094+1.6345)$ = 1.4718	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4+\left\|\sqrt{(1.5253)}\right\|$ = 1.6350	$\frac{1}{2}(1.3094+1.6350)$ = 1.4721	1.2309 + 0.2 (1.4721) = 1.5253
0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6+\left\|\sqrt{(1.8523)}\right\|$ = 1.9610	$\frac{1}{2}(1.635+1.961)$ =1.798	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6+\left\|\sqrt{(1.8849)}\right\|$ = 1.9729	$\frac{1}{2}(1.635+1.9729)$ = 1.8040	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6+\left\|\sqrt{(1.8861)}\right\|$ = 1.9734	$\frac{1}{2}(1.635+1.9734)$ = 1.8042	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

Exercises 10.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out six steps).

2. Using Euler's method, find the approximate value of y when $x = 0.6$ of $dy/dx = 1 - 2xy$, given that $y = 0$ when $x = 0$ (take $h = 0.2$).

3. Using the simple Euler's method solve for y at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.

4. Solve $y' = 1 - y, y(0) = 0$
by the modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$

5. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$, taking $h = 0.05$ by the modified Euler's method.

6. Given $y' = x + \sin y, y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method.

7. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (five steps)

8. Given that $dy / dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method.

10.6 Runge's Method*

Consider the differential equation, $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ (1)

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Figure 10.2).

Integrating both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x, y)dx \qquad (2)$$

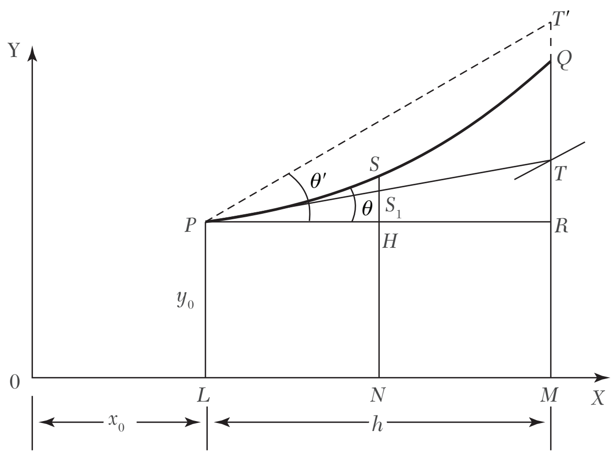

FIGURE 10.2

*Called after the German mathematician *Carl Runge* (1856-1927).

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (*i.e.*, dy/dx) at the points x_0, $x_0 + h/2$, $x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_S is given by the point S_1

$$\therefore \qquad y_s = NS_1 = LP + HS_1 = y_0 + PH.\tan\theta$$

$$= y_0 + h(dy/dx)_p = y_0 + \frac{h}{2} f(x_0, y_0) \tag{3}$$

Also $\qquad y_T = MT = LP + RT = y_0 + PR.\tan\theta = y_0 + hf(x_0 + y_0).$

Now the value of y_Q at $x_0 + h$ is given by the point T'' where the line through P draw with slope at $T(x_0 + h, y_T)$ meets MQ.

$\therefore$ Slope at $T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \qquad y_Q = R + RT = y_0 + PR\ .\tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0,\ y_0)] \tag{4}$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,

the value of $f(x, y)$ at $S = f(x_0 + h/2, y_S)$

and the value of $f(x, y)$ at $Q = (x_0 + h, y_Q)$

where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x,y)dx = \frac{h}{6}\left[f_P + 4f_S + f_Q\right] \qquad \text{by Simpson's rule}$$

$$= \frac{h}{6}\left[f(x_0 + y_0) + f(x_0 + h/2, y_S) + (x_0 + h, y_Q\right]$$

Which gives a sufficiently accurate value of k and also $y = y_0 + k$

The repeated application of (5) gives the values of y for equi-spaced points.

Working rule *to solve* (1) *by Runge's method*:

Calculate successively

$$k_1 = hf(x_0,\ y_0),$$

$$k_2 = hf\left(x_0 + \frac{1}{2}hy_0 + \frac{1}{2}k_1\right)$$

$$k' = hf\left(x_0 + h, y_0 + k_1\right)$$

and $$k_3 = hf\left(x_0 + h, y_0 + k'\right)$$

Finally compute, $k = \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, and k_3).

EXAMPLE 10.14

Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2(1) = 0.200$$

$$k_2 = hf\left(x_0 + \frac{1}{2}hy_0 + \frac{1}{2}k_1\right) = 0.2f(0.1, 1.1) \quad 0.240$$

$$k' = hf\left(x_0 + h, y_0 + k_1\right) = 0.2f(0.2, 1.2) = 0.280$$

and $$k_3 = hf\left(x_0 + h, y_0 + k'\right) = 0.2f(0.1, 1.28) = 0.296$$

$$\therefore \quad k = \frac{1}{6}\left(\mathrm{k}_1 + 4\mathrm{k}_2 + \mathrm{k}_3\right) = \frac{1}{6}(0.200 + 0.960 + 0.296) = 0.2426$$

Hence the required approximate value of y is 1.2426.

10.7 Runge-Kutta Method*

The Taylor's series method of solving differential equations numerically is restricted by the labor involved in finding the higher order derivatives. However, there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the *order of that method*.

First order R-K method. We have seen that Euler's method (Section 10.4) gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because y' = f(x, y)]$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \cdots$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, *Euler's method is the Runge-Kutta method of the first order*.

Second order R-K method. The modified Euler's method gives

$$y_1 = y + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1)\right] \tag{1}$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ on the right-hand side of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h), y_0 + hf_0\right] \quad \text{where} \quad f_0 = (x_0, y_0) \tag{2}$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{3}$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f_0 = (x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)^{**}\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3) \qquad \left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3) \tag{4}$$

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution upto the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

**$O(h^2)$ means "terms containing second and higher powers of h" and is read as *order of* h^2.

∴ *The second order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k)$

(*iii*) *Third order R-K method.* Similarly, it can be seen that Runge's method (Section 10.6) agrees with the Taylor's series solution upto the term in h^3.

As such, *Runge's method is the Runge-Kutta method of the third order.*

∴ *The third order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where, $k_1 = hf(x_0, y_0), k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

And $k_3 = hf\left(x_0 + h, y_0 + k'\right)$, where $k' = k_3 = hf(x_0 + h, y_0 + k_1)$.

(*iv*) *Fourth order R-K method.* This method is most commonly used and is often referred to as the *Runge-Kutta method* only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y), y(x_0)$$

is as follows:

Calculate successively $k_1 = hf(x_0, y_0)$,

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and $$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

Finally compute $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3, and k_4).

NOTE **Obs.** *One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.*

EXAMPLE 10.15

Apply the Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428$$

Hence the required approximate value of y is 1.2428.

EXAMPLE 10.16

Using the Runge-Kutta method of fourth order, solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

Solution:

We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$

Hence $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2 f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196$.

To find $y(0.4)$:

Here $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$.

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$.

EXAMPLE 10.17

Apply the Runge-Kutta method to find the approximate value of y for $x = 0.2$, in steps of 0.1, if $dy/dx = x + y^2$, $y = 1$ where $x = 0$.

Solution:

Given $f(x, y) = x + y^2$.

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x0 = 0$, $y0 = 1$, $h = 0.1$

$\therefore \quad k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) = 0.1347$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$

Step II. $x_1 = x_0 + h = 0.1$, $y_1 = 1.1165$, $h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1f(0.2, 1.1576) = 0.1823$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$

EXAMPLE 10.18

Using the Runge-Kutta method of fourth order, solve for y at $x = 1.2$, 1.4

From $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1$, $y_0 = 0$

Solution:

We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^x}$

To find $y(1.2)$:

Here x_0=1, y_0=0, h=0.2

$$k_1 = hf(x_0, y_0) = 0.2\frac{0+e'}{1+e'} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.073)e^{1+0.1}}{(1+0.1)^2 + (1+0.1)e^{1+0.1}}\right\}$$

$$= 0.1402$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2\left\{\frac{2(1+0.1)(0+0.07)e^{1.1}}{(1+0.1)^2 + (1+0.1)e^{1.1}}\right\}$$

$$= 0.1399$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2\left\{\frac{2(1.2)(0.1399)e^{1.2}}{(1.2)^2 + (1.2)e^{1.2}}\right\}$$

$$= 0.1348$$

and $$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$

$$= 0.1402$$

Hence $y(1.2) = y0 + k = 0 + 0.1402 = 0.1402$.

To find y (1.4):

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2f(1.2, 0) = 0.1348$$

$$k_2 = hf\left(x_1 + h/2, y_1 + k_1/2\right) = 0.2f(1.3, 0.2076) = 0.1303$$

$$k_3 = hf\left(x_1 + h/2, y_1 + k_2/2\right) = 0.2f(1.3, 0.2053) = 0.1301$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(1.3, 0.2703) = 0.1260$$

$\therefore$ $$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260]$$

$$= 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705$.

Exercises 10.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.

2. Using the Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + y^2$, $y(0) = 1$ taking $h = 0.1$.

3. Using the Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\dfrac{dy}{dx} = x^2 + y^2 \;\; y(0) = 1$, taking $h = 0.1$.

4. Use the Runge Kutta method to find y when $x = 1.2$ in steps of 0.1, given that $dy/dx = x^2 + y^2$ and $y(1) = 1.5$.

5. Given $dy/dx = x^3 + y$, $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$, and $y(0.6)$ by the Runge-Kutta method of fourth order.

6. Find $y(0.1)$ and $y(0.2)$ using the Runge-Kutta fourth order formula, given that $y' = x^2 - y$ and $y(0) = 1$.

7. Using fourth order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$. $dy/dx\ (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2.

8. Find by the Runge-Kutta method an approximate value of y for $x = 0.6$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x + y)}$

9. Using the Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$, , $y(0) = 1$. Take $h = 0.2$.

10. Using fourth order Runge-Kutta method, integrate $y' = -2x^3 + 12x^2 - 20x + 8.5$, using a step size of 0.5 and initial condition of $y = 1$ at $x = 0$.

11. Using the fourth order Runge-Kutta method, find y at $x = 0.1$ given that $dy/dx = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$.

12. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$, find y for $x = 0.1$, 0.2, 0.3, 0.4, and 0.5.

10.8 Predictor-Corrector Methods

If x_{i-1} and x_i are two consecutive mesh points, we have $x_i = x_{i-1} + h$. In Euler's method (Section 10.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}h, y_{i-1}); \quad i = 1,2,3\cdots \tag{1}$$

The modified Euler's method (Section 10.5), gives

$$y_i = y_{i-1} + \frac{h}{2}\left[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)\right]$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i. This value of y_i is again substituted in (2) to find a still better approximation of y_i. This step is repeated until two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i.

In the methods so far described to solve a differential equation over an interval, only the value of y at the beginning of the interval was required. In the predictor-corrector methods, four prior values are needed for finding the value of y at x_i. Though slightly complex, these methods have the advantage of giving an estimate of error from successive approximations to y_i.

We now describe two such methods, namely: Milne's method and Adams-Bashforth method.

10.9 Milne's Method

Given $dy/dx = f(x, y)$ and $y = y_0$, $x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows:

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0), f_1 = f(x_0 + h, y_1), f_2 = f(x_0 + 2h, y_2), f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \cdots$$

In the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x,y)dx$$

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots \right) dx$$

[Put $x = x_0 + nh,\ dx = hdn$]

$$= y_0 + \int_0^4 \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots \right) dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \ldots \right)$$

Neglecting fourth and higher order differences and expressing $\Delta f_0, \Delta^2 f_0$ and $\Delta^3 f_0$ and in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

which is called a *predictor*.

Having found y_4, we obtain a first approximation to

$$f_4 = f(x_0 + 4h, y_4)$$

Then a better value of y_4 is found by Simpson's rule as

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$$

which is called a *corrector*.

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged. Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}\left(2f_2 - f_3 + 2f_4\right)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5\right)$$

We repeat this step until y_5 becomes stationary and, then proceed to calculate y_6 as before.

This is *Milne's predictor-corrector method*. To insure greater accuracy, we must first improve the accuracy of the starting values and then subdivide the intervals.

EXAMPLE 10.19

Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$.

Solution:

Using Picard's method, we have

$$y = y(0) + \int_0^x f(x,y)dx, \text{ where } f(x,y) = x - y^2$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

Giving $y_1 = 0 + \int_0^x x dx = \dfrac{x^2}{2}$

To find the second approximation, we put

Giving $y_2 = \int_0^x \left(x - \dfrac{x^4}{4}\right)dx = \dfrac{x^2}{2} - \dfrac{x^5}{20}$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \qquad (i)$$

Now let us determine the starting values of the Milne's method from (i), by choosing $h = 0.2$.

$x_0 = 0.0,\quad y_0 = 0.0000,\quad f_0 = 0.0000$

$x_1 = 0.2,\quad y_1 = 0.020,\quad f_1 = 0.1996$

$x_2 = 0.4,\quad y_2 = 0.0795\quad f_2 = 0.3937$

$x_3 = 0.5,\quad y_3 = 0.1762,\quad f_3 = 0.5689$

Using the predictor, $y_4^{(p)} = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$x = 0.8 \qquad y_4^{(p)} = 0.3049, \qquad f_4 = 0.7070$$

and the corrector, $y_4^{(c)} = y_2 + \dfrac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$$y_4^{(c)} = 0.3046 \qquad f_4 = 0.7072 \qquad (ii)$$

Again using the *corrector*,

$$y_4^{(c)} = 0.3046, \text{, which is the same as in (ii)}$$

Now using the predictor,

$$y_4^{(p)} = y_1 + \frac{4h}{3}\left(2f_2 - f_3 + 2f_4\right)$$

$x = 0.1,$ $\quad y_5^{(p)} = 0.4554 \qquad f_5 = 0.7926$

and the corrector $y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5\right)$ gives

$$y_5^{(c)} = 0.4555 \qquad f_5 = 0.7925$$

Again using the corrector,

$$y_5^{(c)} = 0.4555, \text{ a value which is the same as before.}$$

Hence $y(1) = 0.4555$.

EXAMPLE 10.20

Using Milne's method find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.

Solution:

We have $y' = (2 - y^2)/5x = f(x)$ [say]

Then the starting values of the Milne's method are

$x_0 = 0,$ $\quad y_0 = 1,$ $\quad f_0 = \frac{2-12}{5\times 4} = 0.05$

$x_1 = 4.1,$ $\quad y_1 = 1.0049,$ $\quad f_1 = 0.0485$

$x_2 = 4.2,$ $\quad y_2 = 1.0097,$ $\quad f_2 = 0.0467$

$x_3 = 4.3,$ $\quad y_3 = 1.0143,$ $\quad f_3 = 0.0452$

$x_4 = 4.4,$ $\quad y_4 = 1.0187,$ $\quad f_4 = 0.0437$

Since y_5 is required, we use the predictor

$$y_5^{(p)} = y_1 + \frac{4h}{3}\left(2f_2 - f_3 + 2f_4'\right) \qquad (h = 0.1)$$

$$x = 4.5,\; y_5^{(p)} = 1.0049 + \frac{4(0.1)}{3}\left(2\times 2.0467 - 0.0452 + 2\times 0.0437\right) = 1.023$$

$$f_5 = \frac{2 - y_5^{\,2}}{5x_5} = \frac{2-(1.023)^2}{5\times 4.5} = 0.0424$$

Now using the corrector $y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5\right)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}(0.0452 + 4\times 0.0437 + 0.0424) = 1.023$$

Hence $y(4.5) = 1.023$

EXAMPLE 10.21

Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1$, 0.2, and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.

Solution:

Given $y(0) = 1$ and $h = 0.1$

We have $\quad y'(x) = x(x^2 + y^2)e^{-x} \qquad y'(0) = 0$

$$\therefore \quad y''(x) = \left[(x^3 + xy^2)(e^{-x}) + (3x^2 + y^2 + x(2y)y')\right]e^{-x}$$

$$= e^{-x}\left[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy'\right]; \qquad y''(0) = 1$$

$$y'''(x) = e^{-x}\left[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'\right]$$

$$y'''(0) = 2$$

Substituting these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \cdots$$

$$= 1 + 0.005 - 0.0003 = 1.0047, \textit{ i.e., } 1.005$$

Now taking $\quad x = 0.1$, $y(0.1) = 1.005$, $h = 0.1$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \cdots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{6}(-1247) + \cdots$$

$$= 1.018$$

Now taking $x = 0.2$, $y(0.2) = 1.018$, $h = 0.1$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$y(0.2) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \cdots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001$$
$$= 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0,$	$y_0 = 1$	$f_0 = y_0 = 0$
$x_1 = 0.1,$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2,$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3,$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - 0.176 + 2(0.26)]$$
$$= 1.09.$$

$x = 0.4$ $y_4^{(p)} = 1.09,$ $f_4 = y'(0.4) = 0.362$

Using the corrector, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$$y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $y(0.4) = 1.071$

EXAMPLE 10.22

Using the Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method.

Solution:

We have $f(x, y) = xy + y^2$.

To find $y(0.1)$:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

$\therefore$ $k_1 = hf(x_0, y_0) = (0.1) \times f(0,1) = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) \times f(0.05, 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) \times f(0.05, 1.0577) = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) \times f(0.1, 1.1172) = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$

To find $y(0.2)$:

Here $x_1 = 0.1$, $y_1 = 1.1169$, $h=0.1$

$$k_1 = hf(x_1, y_1) = (0.1) \times f(0.1, 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) \times f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) \times f(0.15, 1.1959) = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) \times f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773.$

To find $y(0.3)$:

Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = hf(x_2, y_2) = (0.1) \times f(0.2, 1.2773) = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) \times f(0.25, 1.3716) = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 1.3885) = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 1.5048) = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2267$$

$$y(0.3) = y_3 = y_2 + k = 1.504$$

Now the starting values for the Milne's method are:

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$x_4 = 0.4 \quad y_4^{(p)} = 1.8344 \quad f_4 = 4.0988$$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8397 \qquad f_4 = 4.1159.$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$

$$= 1.8391 \qquad f_4 = 4.1182 \qquad \text{(i)}$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$

$$= 1.8392 \text{ which is same as (i)}$$

Hence $y(0.4) = 1.8392$.

Exercises 10.4

1. Given $\frac{dy}{dx} = x^3 + y,\ y(0) = 2$. The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are gotten by the R.K. method of the order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$

2. Given $2\,dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor corrector method.
3. Solve that initial value problem

$$\frac{dy}{dx} = 1 + xy^2, y(0) = 1$$

for $x = 0.4$ by using Milne's method, when it is given that

x:	0.1	0.2	0.3
y:	1.105	1.223	1.355

4. From the data given below, find y at $x = 1.4$, using Milne's predictor-corrector formula: $dy/dx = x^2 + y/2$:

$x = 1$	1.1	1.2	1.3
$y = 2$	2.2156	2.4549	2.7514

5. Using Taylor's series method, solve $\frac{dy}{dx} = xy + x^2,\; y(0) = 1$; at $x = 0.1$, 0.2, 0.3. Continue the solution at $x = 0.4$ by Milne's predictor-corrector method.
6. If $y = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.01$, $y(0.2) = 2.04$, and $y = 2.09$, find $y(0.4)$ using Milne's predictor-corrector method.
7. Using the Runge-Kutta method, calculate y (0.1), $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} = \frac{2xy}{1+x^2} = 1.\; y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

10.10 Adams-Bashforth Method

Given $\frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h), y_{-2} = y(x_0 - 2h), y_{-3} = y(x_0 - 3h)$$

by Taylor's series or Euler's method or the Runge-Kutta method.

Next we calculate

$$f_{-1} = f(x_0 - h, y_{-1}), f_{-2} = f(x_0 - 2h, y_{-2}), f_{-3} = f(x_0 - 3h, y_{-3})$$

Then to find y_1, we substitute Newton's backward interpolation formula

$$f(x, y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2}{6}\nabla^3 f_0 + \cdots$$

in $$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x,y) \tag{1}$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots\right)dx$$

[Put $x = x_0 + nh$, $dx = hdn$]

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots\right)dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1 = y_0 + \frac{h}{24}\left(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right) \tag{2}$$

This is called the *Adams-Bashforth predictor formula*.

Having found y_1, we find $f_1 = f(x_0 + h_1, y_1)$.

Then to find a better value of $y1$, we derive a *corrector formula* by substituting Newton's backward formula at $f1$, *i.e.*,

$$f(x,y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2}{6}\nabla^3 f_1 + \cdots$$

in (1)

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right)dx$$

[Put $x = x_1 + nh$, $dx = h\,dn$]

$$= y_0 + \int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right)dn$$

$$= y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_0 - \frac{1}{24}\nabla^3 f_1 + \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ and in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24}\left(9 f_1 + 19 f_0 - 5 f_{-1} + 9 f_{-2}\right)$$

which is called the *Adams-Moulton corrector formula*.

Then an improved value of $f1$ is calculated and again the corrector (3) is applied to find a still better value $y1$. This step is repeated until y_1 remains unchanged and then we proceed to calculate y_2 as above.

NOTE **Obs.** *To apply both Milne and Adams-Bashforth methods, we require four starting values of y which are calculated by means of Picard's method or Taylor's series method or Euler's method or the Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with the fourth order Runge-Kutta formulae have been found to be the most useful.*

EXAMPLE 10.23

Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by the Adams-Bashforth method.

Solution:

Here $f(x, y) = x^2(1 + y)$

Starting values of the Adams-Bashforth method with $h = 0.1$ are

$x = 1.0,\quad y_{-3} = 1.000,\quad f_{-3} = (1.0)^2(1 + 1.000) = 2.000$

$x = 1.1,\quad y_{-2} = 1.233,\quad f_{-2} = 2.702$

$x = 1.2,\quad y_{-1} = 1.548,\quad f_{-1} = 3.669$

$x = 1.3,\quad y_0 = 1.979,\quad f_0 = 5.035$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}\left(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right)$$

$$x_4 = 1.4, \quad y_1^{(p)} = 2.573 \quad f_1 = 7.004$$

Using the *corrector*

$$y_1^{(c)} = y_0 + \frac{h}{24}\left(9f_1 + 19f_0 - 5f_{-1} + f_{-2}\right)$$

$$y_1^{(c)} = 1.979 + \frac{0.1}{24}(9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$$

Hence $y(1.4) = 2.575$

EXAMPLE 10.24

If $\dfrac{dy}{dx} = 2e^x y,\ y(0) = 2,$ find $y(4)$ using the Adams predictor corrector formula by calculating $y(1)$, $y(2)$, and $y(3)$ using Euler's modified formula.

Solution:

We have $f(x, y) = 2e^x y$

x	$2e^x y$	*Mean slope*	*Oldy + h (mean slop) = new y*
0	4		$2 + 0.1(4) = 2.4$
0.1	$2e^{0.1}(2.4) = 5.305$	$\frac{1}{2}(4+5.305) = 4.6524$	$2 + 0.1\ (4.6524) = 2.465$
0.1	$2e^{0.1}(2.465) = 5.449$	$\frac{1}{2}(4+5.449) = 4.7244$	$2 + 0.1\ (4.7244) = 2.472$
0.1	$2e^{0.1}(2.4724) = 5.465$	$\frac{1}{2}(4+5.465) = 4.7324$	$2 + 0.1\ (4.7324) = 2.473$
0.1	$2e^{0.1}(2.478) = 5.467$	$\frac{1}{2}(4+5.467) = 4.7333$	$2 + 0.1\ (4.7333) = 2.473$
0.1	5.467	—	$2 + 0.1\ (5.467) = 3.0199$
0.2	$2e^{0.2}(3.0199) = 7.377$	$\frac{1}{2}(5.467+7.377) = 6.422$	$2.473 + 0.1\ (6.422) = 3.1155$
0.2	7.611	$\frac{1}{2}(5.467+7.611) = 6.539$	$2.473 + 0.1\ (6.539) = 3.127$
0.2	7.639	$\frac{1}{2}(5.467+7.639) = 6.553$	$2.473 + 0.1\ (6.553) = 3.129$
0.2	7.643	$\frac{1}{2}(5.467+7.643) = 6.555$	$2.473 + 0.1\ (6.555) = 3.129$
0.2	7.463	—	$3.129 + 0.1\ (7.643) = 3.893$
0.3	$2e^{0.3}(3.893) = 10.51$	$\frac{1}{2}(7.643+10.51) = 9.076$	$3.129 + 0.1\ (9.076) = 4.036$
0.3	10.897	$\frac{1}{2}(7.643+10.897) = 9.266$	$3.129 + 0.1\ (9.2696) = 4.056$
0.3	10.949	$\frac{1}{2}(7.643+10.949) = 9.296$	$3.129 + 0.1\ (9.296) = 4.058$
0.3	10.956	$\frac{1}{2}(7.643+10.956) = 9.299$	$3.129 + 0.1\ (9.299) = 4.0586$

To find $y(0.4)$ *by Adam's method*, the starting values with $h = 0.1$ are

$x = 0.0 \qquad y_{-3} = 2.4 \qquad f_{-3} = 4$

$x = 0.1 \qquad y_{-2} = 2.473 \qquad f_{-2} = 5.467$

$x = 0.2 \qquad y_{-1} = 3.129 \qquad f_{-1} = 7.643$

$x = 0.3 \qquad y_{0} = 4.059 \qquad f_{0} = 10.956$

Using the predictor formula

$$y_1^{(p)} = y_0 + \frac{h}{24}\left(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right)$$

$$= 4.059 + \frac{0.1}{24}(55\times 10.957 - 59\times 7.643 + 37\times 5.467 - 9\times 4)$$

$$= 5.383$$

Now $\quad x = 0.4 \quad y_1 = 5.383 \quad f_1 = 2e^{0.4}(5.383) = 16.061$

Using the corrector formula

$$y_1^{(c)} = y_0 + \frac{h}{24}\left(9f_1 + 19f_0 - 5f_{-1} + f_{-2}\right)$$

$$= 4.0586 + \frac{0.1}{24}(9\times 16.061 + 19\times 10.956 - 5\times 7.643 + 5.467)$$

$$= 5.392$$

Hence $y(0.4) = 5.392$

EXAMPLE 10.25

Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using the Runge-Kutta method of the fourth order using step value $h = 0.1$

Solution:

We have $f(x, y) = x - y^2$.

To find $y(0.1)$:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

$\therefore$

$$k_1 = hf(x_0, y_0) = (0.1)\,f(0,1) = -0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 0.95) = -0.08525$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 0.9574) = -0.0867$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = (0.1)\,f(0.1, 0.9137) = -0.07341$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)0 = -0.0883$$

Thus $y(0.1) = y_1 = y_0 + k = 1 - 0.0883 = 0.9117$

To find $y(0.2)$:

Here $x_1 = 0.1$, $y_1 = 0.9117$, $h = 0.1$

$$k_1 = hf(x_1, y_1) = (0.1) \times f(0.1, 0.9117) = -0.0731$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\, f(0.15, 0.8751) = 0.0616$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\, f(0.15, 0.8809) = 0.0626$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) \times f(0.2, 0.8491) = 0.0521$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0623$$

Thus $y(0.2) = y_2 = y_1 + k = 0.8494.$

To find $y(0.3)$:

Here $x_2 = 0.2$, $y_2 = 0.8494$, $h = 0.1.$

$$k_1 = hf(x_2, y_2) = (0.1) \times f(0.25, 0.8494) = 0.0521$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\, f(0.25, 0.8233) = 0.0428$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\, f(0.25, 0.828) = 0.0436$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\, f(0.3, 0.058) = 0.0349$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0438$$

Thus $y(0.3) = y_3 = y_2 + k = 0.8061$

Now the starting values for the Milne's method are:

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 0.0 - (0.1)^2 = 1.0000$
$x_1 = 0.1$	$y_1 = 0.9117$	$f_1 = 0.1 - (0.9117)^2 = -0.7312$
$x_2 = 0.2$	$y_2 = 0.8494$	$f_2 = 0.2 - (0.8494)^2 = -0.5215$
$x_3 = 0.3$	$y_3 = 0.8061$	$f_3 = 0.3 - (0.8061)^2 = -0.3498$

Using the predictor,

$$y_1^{(p)} = y_0 + \frac{h}{24}\left(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right)$$

$x = 0.4$

$$y_1^{(p)} = 0.8061 + \frac{0.1}{24}\left(55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1)\right)$$

$$= 0.7789 \quad f_1 = -0.2.67$$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}\left(9f_1 + 19f_0 - 5f_{-1} + f_{-2}\right)$$

$$y_1^{(c)} = 0.8061 + \frac{0.1}{24}\left[9(-0.2067) + 19\times(-0.3498) - 5(-0.5215) - 0.7312\right]$$

$$= 0.7785$$

Hence $y(0.4) = 0.7785$

Exercises 10.5

1. Using the Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

x:	0	0.2	0.4	0.6
y:	0	0.0200	0.0795	0.1762

2. Using the Adams-Bashforth formulae, determine $y(0.4)$ given the differential equation $dy/dx = \frac{1}{2}xy$ and the data:

x:	0	0.1	0.2	0.3
y:	1	1.002	5 1.0101	1.0228

3. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0.4)$ using the Adams-Bashforth method.

4. Using the Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

5. Given the differential equation $dy/dx = x^2y + x^2$ and the data:

x:	1	1.1	1.2	1.3
y:	1	1.233	1.548488	1.978921

determine $y(1.4)$ by any numerical method.

6. Using the Adams-Bashforth method, evaluate $y(1.4)$; if y satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$.

10.11 Simultaneous First Order Differential Equations

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{1}$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \tag{2}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially Picard's or Runge-Kutta methods.

Picard's method gives

$$y_1 = y_0 + \int f(x, y_0, z_0)dx,\ z_1 = z_0 + \int \phi(x, y_0, z_0)dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)dx,\ z_2 = z_0 + \int \phi(x, y_1, z_1)dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)dx,\ z_3 = z_0 + \int \phi(x, y_2, z_2)dx$$

and so on.

(*ii*) *Taylor's series method* is used as follows:

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0''' + \frac{h^3}{3!}y_0''' + \cdots \tag{3}$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \tag{4}$$

Differentiating (1) and (2) successively, we get y'', z'', etc. So the values $y_0', y_0'', y_0''' \cdots$ and $z_0, z_0'', z_0''' \cdots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1'' + \cdots \tag{5}$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \tag{6}$$

Since y_1 and z_1 are known, we can calculate $y_1', y_1'', \cdots$ and $z_1', z_1'', \cdots$. Substituting these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

(*iii*) *Runge-Kutta method* is applied as follows:

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0)$$

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3, z_0 + \frac{1}{2}l_3\right)$$

$$l_4 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3, z_0 + \frac{1}{2}l_3\right)$$

Hence $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $$z_1 = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

EXAMPLE 10.26

Using Picard's method, find approximate values of y and z corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and $dy/dx = x + z$, $dz/dx = x - y^2$.

Solution:

Here $x_0 = 0$, $y_0 = 2$, $z_0 = 1$,

and $$\frac{dy}{dx} = f(x,y,z) = x + z$$

$$\frac{dz}{dx} = \phi(x,y,z) = x - y^2$$

$$\therefore \qquad y = y_0 + \int_{x_0}^{x} f(x,y,z)dx \text{ and } z = z_0 + \int_{x_0}^{x} \phi(x,y,z)dx$$

First approximations

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0,z_0)dx = 2 + \int_{x_0}^{x} (x+1)dx = 2 + x + \frac{1}{2}x^2$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x,y_0,z_0)dx = 1 + \int_{x_0}^{x} (x-4)dx = 1 - 4x + \frac{1}{2}x^2$$

Second approximations

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1,z_1)dx = 2 + \int_{0}^{x} (1 - 4x + \frac{1}{2}x^2)dx$$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x,y_1,z_1)dx = 1 + \int_{x_0}^{x} \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right]dx$$

$$= 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}$$

Third approximations

$$y_3 = y_0 + \int_{x_0}^{x} f(x,y_2,z_2)dx = 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x,y_2,z_2)dx$$

$$= 1 - 4x + \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$

$$y_1 = 2.105, \qquad y_2 = 2.08517, \qquad y_3 = 2.08447$$

$$z_1 = 0.605, \qquad z_2 = 0.58397, \qquad z_3 = 0.58672.$$

Hence $y(0.1) = 2.0845, \qquad z(0.1) = 0.5867$

correct to four decimal places.

EXAMPLE 10.27

Find an approximate series solution of the simultaneous equations $dx/dt = xy + 2t$, $dy/dt = 2ty + x$ subject to the initial conditions $x = 1$, $y = -1$, $t = 0$.

Solution:

x and y both being functions of t, Taylor's series gives

and
$$\left.\begin{aligned} x(t) &= x_0 + tx_0' + \frac{t^2}{2!}x_0'' + \frac{t^3}{3!}x_0''' + \cdots \\ y(t) &= y_0 + ty_0' + \frac{t^2}{2!}y_0'' + \frac{t^3}{3!}y_0''' + \cdots \end{aligned}\right\} \qquad (i)$$

Differentiating the given equations

$$x' = xy + 2t \qquad (ii)$$

$$y' = 2ty + x \qquad (iii)$$

w.r.t. t, we get

$$\begin{aligned} x'' &= xy' + x'y + 2 \\ x''' &= (xy'' + x'y') + x''y + x'y' \end{aligned} \quad \left.\begin{aligned} y'' &= 2ty' + 2y + x' \\ y''' &= 2ty'' + 2y' + 2y' + x'' \end{aligned}\right\} \qquad \text{(iv)}$$

Putting $x_0 = 1$, $y_0 = -1$, $t_0 = 0$ in (ii), (iii), and (iv), we obtain

$$\begin{aligned} x_0 &= -1 + 2(0) = -1 \\ x_0' &= x_0 y_0' + x_0' y_0 + 2 \\ &= 1.1 + (-1)(-1) + 2 = 4 \\ x_0''' &= -3 + (-1)(1) + 4(-1) - 1 = -9 \end{aligned} \quad \begin{aligned} y_0' &= 1 \\ y_0''' &= 0 + 2y_0 + x_0' \\ &= 2(-1) + (-1) = -3 \\ y_0''' &= 2 + 2 + 4 = 8 \text{ etc} \end{aligned}$$

Substituting these values in (i), we get

$$x(t) = 1 - t + 4\frac{t^2}{2!} + (-9)\frac{t^3}{3!} + \cdots = 1 - t + 2t^2 - \frac{3}{2}t^3 + \cdots$$

$$x(t) = 1 + t + 3\frac{t^2}{2!} + 8\frac{t^3}{3!} + \cdots = 1 + t - \frac{3}{2}t^2 + \frac{4}{3}t^3 + \cdots$$

EXAMPLE 10.28

Solve the differential equations

$$\frac{dy}{dx} = 1 + xz, \frac{dz}{dx} = -xy \text{ for } x = 0.3$$

using the fourth order Runge-Kuta method. Intial values are $x = 0$, $y = 0$, $z = 1$.

Solution:

Here $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$

$x_0 = 0$, $y_0 = 0$, $z_0 = 1$. Let us take $h = 0.3$.

$$\therefore \quad k_1 = hf(x_0, y_0, z_0) = 0.3f(0, 0, 1) = 0.3(1 + 0) = 0.3.$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.3(-0 \times 0) = 0$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3)f(0.15, 0.15, 1) = 0.3(1 + 0.15) = 0.345$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3)[-(0.15)(0.15)] = -0.00675$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.3)f(0.15, 0.1725, 0.996625)$$

$$= 0.3[1 + 0.996625 \times 0.15] = 0.34485$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.3[-(0.15)(0.1725)] = -0.007762$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3[-(0.3)(0.34485)] = -0.03104$$

Hence $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

i.e., $y(0.3) = 0 + \frac{1}{6}[0.3 + 2(0.345) + 2(0.34485) + 0.3893] = 0.34483$

and $z(x+h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

i.e. $z(0.3) = 1 + \frac{1}{6}[0 + 2(-0.00675) + 2(0.0077625) + (-0.03104)]$

$= 0.98999$

10.12 Second Order Differential Equations

Consider the second order differential equation

$$\frac{d^2y}{dx_2} = f\left(x, y, \frac{dy}{dx}\right)$$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential Equations

$$\frac{dy}{dx} = z, \frac{dz}{dx}\text{f(x, y, z)}$$

These equations can be solved as explained above.

EXAMPLE 10.29

Find the value of $y(1.1)$ and $y(1.2)$ from $y'' + y^2y' = x3; y(1) = 1, y'(1) = 1$, using the Taylor series method

Solution:

Let $y' = z$ so that $y'' = z'$

Then the given equation becomes $z' + y^2z = z^3$

$\therefore$ $y' = z$

$z' = x^3 - y^2z$ (*i*)

such that $y(1) = 1, z(1) = 1, h = 0.1.$ (*ii*)

Now from (i) $y' = z, y'' = z', y''' = z''$ (*iii*)

$$\left.\begin{aligned} z' &= x^3 - y^2z, z'' = 3x^2 - y^2z' - 2yz^2 \left(\because y' = z\right) \\ \text{and from (ii)} \quad z''' &= 6x - (y^2z'' + 2yy'z') - 2\left(y'z^2 + y^2zz'\right) \\ &= 6x - (y^2z'' + 2yz'^2) - 2\left(z^3 + 2yzz'\right) \end{aligned}\right\} \quad (iv)$$

Taylor's series for $y(1.1)$ is

$$y(1.1) = y(1) + hy'(1) + \frac{h^2}{2!}y''(1) + \frac{h^3}{3!}y'''(1) + ...$$

Also $\quad y(1) = 1, y'(1) = 1, y''(1) = z'(1) = 0, y'''(1) = z''(1) = 1$

$$\therefore \quad y(1.1) = (1) + 0.1(1) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(0) = 1.1002.$$

Taylor's series for $z(1.1)$ is

$$z(1.1) = z(1) + hz'(1) + \frac{h^2}{2!}z''(1) + \frac{h^3}{3!}z'''(1) + ...$$

Here $\quad z(1) = 1, z'(1) = 0, z''(1) = 1, z'''(1) = 3$

$$z(1.1) = (1) + 0.1(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(3) = 1.0055$$

Hence $y(1.1) = 1.1002$ and $z(1.1) = 1.0055$.

EXAMPLE 10.30

Using the Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.

Solution:

Let $dy/dx = z = f(x, y, z)$

Then $\quad \dfrac{dy}{dx} = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0$, $y_0 = 1$, $z_0 = 0$, $h = 0.2$

$\therefore$ Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0$$

$$k_2 = hf\left(x_o + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.2(-0.1) = -0.02$$

$$k_3 = hf\left(x_o + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$= 0.2(-0.0999) = -0.02$$

$$k_4 = hf\left(x_o + h, y_0 + k_3, z_0 + l_3\right)$$

$$= 0.2(-0.1958) = -0.0392$$

$$\therefore \qquad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0199$$

$$l_1 = hf(x_0, y_0, z0) = 0.2(-1) = -0.2$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.2(-0.999) = -0.1998$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$= 0.2(-0.9791) = -0.1958$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.2(0.9527) = -0.1905$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $$y' = z = z0 + l = 0 - 0.1970 = -0.1970.$$

EXAMPLE 10.31

Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, obtain y for $x = 0(0.1)\,0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.

Solution:

Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, y' = z \tag{1}$$

We employ Taylor's series method to find y.

Differentiating the given equation n times, we get

$$y_{n+2} + x_{n+1} + ny_n + y_n = 0$$

At $\quad x = 0$, $(y_{n+2})_0 = -(n+1)(y_n)_0$

$\therefore \quad y(0) = 1$, gives $y_2(0) = -1$, $y_4(0) = 3$, $y_6(0) = -5 \times 3$, ...

and $\quad y_1(0) = 0$ yields $y_3(0) = y_5(0) = \ldots\ldots = 0$.

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots$$

$$\therefore \qquad y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{5\times 3}{6!}x^6 + \cdots \qquad (2)$$

and $$z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots = -xy, \qquad (3)$$

From (2), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - \cdots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \cdots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48}\cdots = 0.956$$

From (3), we have

$$z(0.1) = -0.0995,\ z(0.2) = -0.196,\ z(0.3) = -0.2863.$$

Also from (1), $z'(x) = -(xz + y)$

$$\therefore \qquad z'(0.1) = 0.985,\ z'(0.2) = -0.941,\ z'(0.3) = -0.87.$$

Applying Milne's predictor formula, first to z and then to y, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692$$

and $$y(0.4) = y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\} \quad [\because y' = z]$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231$$

Also $$z'(0.4) = -\{x(0.4)\,z(0.4) + y(0.4)\}$$

$$= -\{0.4(-0.3692) + 0.9231\} = -0.7754.$$

Now applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\}$$

$$= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692$$

and $$y(0.4) = y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\}$$

$$= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232$$

Hence $y(0.4) = 0.9232$ and $z(0.4) = -0.3692$.

Exercises 10.6

1. Apply Picard's method to find the third approximations to the values of y and z, given that

$dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1$, $z = 1/2$ when $x = 0$.

2. Using Taylor's series method, find the values of x and y for $t = 0.4$, satisfying the differential equations

$dx/dt = x + y + t$, $d^2y/dt^2 = x - t$ with initial conditions $x = 0$, $y = 1$, $dy/dt = -1$ at $t = 0$.

3. Solve the following simultaneous differential equations, using Taylor series method of the fourth order, for $x = 0.1$ and 0.2:

$$\frac{dy}{dx} = xz + 1; \frac{dz}{dy} == xy; y(0) = 1.$$

4. Find $y(0.1)$, $z(0.1)$, $y(0.2)$, and $z(0.2)$ from the system of equations: $y' = x + z$, $z' = x - y^2$ given $y(0) = 0$, $z(0) = 1$ using Runge-Kutta method of the fourth order.

5. Using Picard's method, obtain the second approximation to the solution of

$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y \text{ so that } y(0) = 1.y'(0) = \frac{1}{2}.$$

6. Use Picard's method to approximate y when $x = 0.1$, given that

$$\frac{d2y}{dx2} + 2x\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0 \text{ and } y = 0.5, \frac{dy}{dx = 0.1} \text{ when } x = 0.$$

7. Find $y(0.2)$ from the differential equation $y'' + 3xy' - 6y = 0$ where $y(0) = 1$, $y'(0) = 0.1$, using the Taylor series method.

8. Using the Runge-Kutta method of the fourth, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.

9. Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t} \sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runga-Kutta method, find $y(0.2)$.

10. The angular displacement θ of a simple pendulum is given by the equation

$$\frac{d^2\phi}{dt^2} + \frac{g}{1}\sin\phi = 0$$

where $l = 98$ cm and g = 980 cm/sec^2. If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use the Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.

11. In a L-R-C circuit the voltage $v(t)$ across the capacitor is given by the equation

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = 0$$

subject to the conditions $t = 0$, $v = v_0$, $dv/dt = 0$.
Taking $h = 0.02$ sec, use the Runge-Kutta method to calculate v and dv/dt when $t = 0.02$, for the data $v_0 = 10$ volts, $C = 0.1$ farad, $L = 0.5$ henry and $R = 10$ ohms.

10.13 Error Analysis

The numerical solutions of differential equations certainly differ from their exact solutions. *The difference between the computed value yi and the true value y(xi) at any stage is known as the* **total error.** *The total error at any stage is comprised of* **truncation error** *and* **round-off error.**

The most important aspect of numerical methods is to minimize the errors and obtain the solutions with the least errors. It is usually not possible to follow error development quite closely. We can make only rough estimates. That is why, our treatment of error analysis at times, has to be somewhat intuitive.

In any method, the truncation error can be reduced by taking smaller sub-intervals. The round-off error cannot be controlled easily unless the computer used has the double precision arithmetic facility. In fact, this error has proved to be more elusive than the truncation error.

The truncation error in Euler's method is $\frac{1}{2}h^2 yn'''$, i.e., of (h^2) while that of modified Euler's method is $\frac{1}{2}h^3 y_{n'''}$, .i.e., of (h^3)

Similarly in the fourth order of the Runge-Kutta method, the truncation error is of $O(h^5)$.

In the Milne's method, the truncation error

due to predictor formula $= \frac{14}{45} y_n^v h^5$

and due to corrector formula $= -\frac{1}{90} y_n^v h^5$.

i.e., the truncation error in Milne's method is also of $O(h^5)$.

Similarly the error in the Adams-Bashforth method is of the fifth order. Also the predictor error T_p and the corrector error T_c are so related that $19T_p \approx -251\, T_c$.

The **relative error** *of an approximate solution is the ratio of the total error to the exact value*. It is of greater importance than the error itself for if the true value becomes larger, then a larger error may be acceptable. If the true value diminishes, then the error must also diminish otherwise the computed results may be absurd.

EXAMPLE 10.32

Does applying Euler's method to the differential equation

$dy/dx = f(x, y)$, $y(x_0) = y_0$, estimate the total error?

When $f(x, y) = -y$, $y(0) = 1$, compute this error neglecting the round-off error.

Solution:

We know that Euler's solution of the given differential equation is

$$y_{n+1} = y_n + hf(x_n, y_n) \text{ where } x_n = x_0 + nh.$$

i.e., $$y_{n+1} = y_n + hy_n' \qquad (1)$$

Denoting the exact solution of the given equation at $x = x_n$ by $y(x_n)$ and expanding $y(x_{n+1})$ by Taylor's series, we obtain

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}, y''(\xi_n), x_n \le \xi_n \le x_n + 1 \qquad (2)$$

$\therefore$ The truncation error $T_{n+1} = y(x_{n+1}) - y_{n+1} = (1/2)h^2\, y''(\zeta_n)$

Thus the truncation error is of $O(h^2)$ as $h \to 0$.

To include the effect of round-off error R_n, we introduce a new approximation y_n which is defined by the same procedure allowing for the round-off error.

$$\overline{y}_{n+1} = \overline{y}_n + hf\left(x_n, \overline{y}_n\right) - R_{n+1} \tag{3}$$

∴ The total error is defined by

$$E_{n+1} = y(x_{n+1}) - \overline{y}_{n+1} \qquad [(2) - (3)]$$

$$= y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''\left(\xi_n\right) - \left\{\overline{y}_n + hf(x_n, y_n) - R_{n+1}\right\}$$

$$= \left[y(x_n) - \overline{y}_n\right] + h\left[h'(x_n) - f(x_n, y_n)\right] + T_{n+1} + R_{n+1} \tag{4}$$

Assuming continuity of $\partial f/\partial y$ and using Mean-Value theorem, we have

$f[x_n, y(x_n)] - f(x_n, y_n) = [y(x_n) - y_n] fy(x_n, \zeta_n)$, where ζ_n lies between $y(x_n)$ and y_n.

∴ (4) takes the form

$$E_{n+1} = [y(x_n) - \overline{y}_n\left[1 + hf_y\left(x_n, \xi_n\right)\right] + T_{n+1} + R_{n+1}$$

or $$E_{n+1} = E_n\,[1 + hf_y(x_n, \zeta_n)] + T_{n+1} + R_{n+1} \tag{5}$$

This is the *recurrence formula* for finding the total error. The first terms on the right-hand side is the *inherited error, i.e.,* the propagation of the error from the previous step y_n to y_{n+1}.

(*b*) We have $dy/dx = -y$, $y(0) = 1$.

Taking $h = 0.01$ and applying (1) successively, we obtain

$$y(0.01) = 1 + 0.01(-1) = 0.99$$

$$y(0.02) = 0.99 + 0.01\,(-0.99) = 0.9801$$

$$y(0.03) = 0.9703,\ y(0.04) = 0.9606$$

∴ The truncation error

$T_{n+1} = (1/2)h_2 y''(\xi) = 0.00005y\xi) \leq 5 \times 10^{-5}\, y(x_n)$ [∵ dy/dx is – ve]

i.e., $$T_1 \leq 5 \times 10^{-5}\, y(0) = 5 \times 10^{-5}$$

$$T_2 \leq 5 \times 10^{-5}\, y(0.01) = 5 \times 10^{-5}\,(0.99) < 5 \times 10{-}5$$

$$T_3 \leq 5 \times 10^{-5}\, y(0.02) = 5 \times 10^{-5}\,(0.9801) < 5 \times 10{-}5$$

$$T_4 \leq 5 \times 10^{-5}\, y(0.03) = 5 \times 10^{-5}\,(0.9703) < 5 \times 10{-}5 \text{ etc.}$$

Also $1 + hf_0(x_n, y_n) = 1 + 0.01(-1) = 0.99$.

Neglecting the round-off error and using the above results, (5) gives

$$E_0 = 0,\ E_1 = E0(0.99) + T_1 \leq 5 \times 10^{-5} = 0.00005$$

$$E_2 = E_1(0.99) + T_2 < 5 \times 10\text{–}5 + 5 \times 10^{-5} = 0.0001$$

$$E_3 = E_2(0.99) + T_3 < 10\text{–}4 + 5 \times 10^{-5} = 0.00015$$

$$E_4 = E_3(0.99) + T_4 < 1.5 \times 10\text{–}4 + 5 \times 10^{-5} = 0.0002 \text{ etc.}$$

NOTE **Obs.** *The exact solution is* $y = e^{-x}$.

$\therefore$ *Actual error in* $y(0.03) = e^{-0.03} - 0.9703 = 0.00014$

and actual error in $y(0.04) = e^{-0.04} - 0.9606 = 0.00019$.

Clearly the total error E_4 *agrees with the actual error in* $y(0.04)$.

10.14 Convergence of a Method

Any *numerical method for solving a differential equation is said to be convergent if the approximate solution* y_n *approaches the exact solution* $y(x_n)$ *as h tends to zero provided the rounding errors arise from the initial conditions approach zero*. This means that as a method is continually refined by taking smaller and smaller step-sizes, the sequence of approximate solutions must converge to the exact solution.

Taylor's series method is convergent provided $f(x, y)$ possesses enough continuous derivatives. The Runge-Kutta methods are also convergent under similar conditions. Predictor corrector methods are convergent if $f(x, y)$ satisfies the *Lipschitz condition, i.e.*,

$$| f(x,y) - f(x,\overline{y}) | \leq k | y - \overline{y} |,$$

k being a constant, then the sequence of approximations to the numerical solution converges to the exact solution.

10.15 Stability Analysis

There is a limit to which the step-size h can be reduced for controlling the truncation error, beyond which a further reduction in h will result in the increase of round-off error and hence increase in the total error. This behavior of the error bound is shown in Figure 10.3.

In such situations, we have to use stable methods so that an error introduced at any stage does not get magnified.

A method is said to be **stable** *if it produces a bounded solution which imitates the exact solution. Otherwise it is said to be* **unstable.** If a method is stable for all values of the parameter, it is said to be *absolutely* or *unconditionally stable*. If it is stable for some values of the parameter, it is said to be *conditionally stable*.

The Taylor's method and Adams-Bashforth method prove to be relatively stable. Euler's method and the Runge-Kutta method are conditionally stable as will be seen from Example 10.23.

The Milne's method is however, unstable since when the parameter is negative, each of the errors is magnified while the exact solution decays.

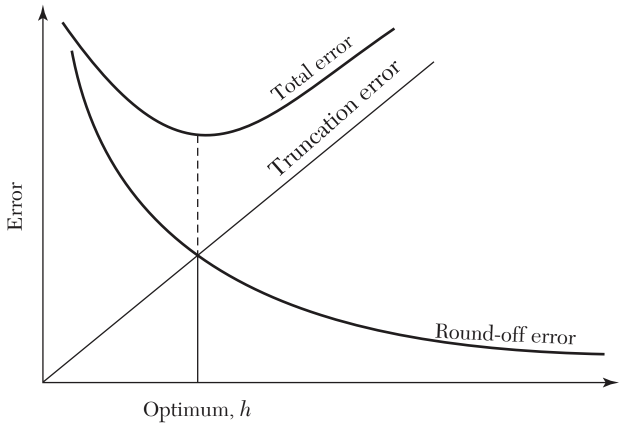

FIGURE 10.3

EXAMPLE 10.33

Does applying Euler's method to the equation

dy/dx = λy, *given* $y(x_0) = y_0$,

determine its stability zone? What would be the range of stability when $\lambda = -1$?

Solution:

We have $y' = \lambda y,\ y(x_0) = y_0$ (1)

By Euler's method,

$$y_n = y_{n-1} + hy'_{n-1} = y_{n-1} + \lambda h y_{n-1} = (1 + \lambda h)y_{n-1} \quad \text{[by (1)}$$

$\therefore \qquad y_{n-1} = (1 + \lambda h)\, y_{n-2}$

....

$y_2 = (1 + \lambda h)\, y_1$

$y_1 = (1 + \lambda h)\, y_0$

Multiplying all these equations, we obtain

$$y_n = (1 + \lambda h)^n\, y_0 \qquad (2)$$

Integrating (1), we get $y = ce^{\lambda x}$

Using $y(x_0) = y_0, y_0 = ce^{\lambda x0} \qquad \therefore y = y_0 e^{l(x-x0)}$

In particular, the exact solution through (x_n, y_n) is

$$y_n = y_0 e^{\lambda(xn - x0)} = y_0 e^{\lambda nh} \qquad [\because x_n = x_0 + nh]$$

or
$$y_n = y_0 (e^{\lambda h})^n = y_0 \left[1 + \lambda h + \frac{(\lambda h)^2}{2} + \cdots\right]^n \qquad (3)$$

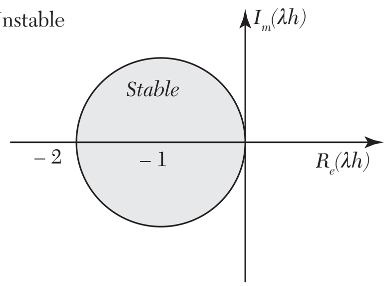

FIGURE 10.4

Clearly the numerical solution (2) agrees with exact solution (3) for small values of h. The solution (2) increases if $|1 + \lambda h| > 1$.

Hence $|1 + \lambda h| < 1$ defines a stable zone.

When λ is real, then the method is stable if $|1 + \lambda h| < 1$ *i.e.* $-2 < \lambda h < 0$

When λ is complex $(= a + ib)$, then it is stable if

$|1 + (a + ib)\, h\, | < 1$ *i.e.* $(1 + ah)^2 + (bh)^2 < 1$

i.e., $(x + 1)2 + y_2 < 1, \qquad$ [where $x = ah, y = bh$.]

i.e., λh lies within the unit circle shown in Figure 10.4.

When λ is imaginary $(= ib)$, $|1 + \lambda h| = 1$, then we have a *periodic-stability*.

Hence Euler's method is absolutely stable if and only if

(i) real λ: $-2 < \lambda h = 0$.

(ii) complex λ: λh lies within the unit circle (Figure 10.4), *i.e.*, Euler's method is conditionally convergent.

When $\lambda = -1$, the solution is stable in the range $-2 < -h < 0$ *i.e.* $0 < h < 2$.

Exercises 10.7

1. Show that the approximate values y_i, obtained from $y' = y$ with $y(0) = 1$ by Taylor's series method, converge to the exact solution for h tending to zero.

2. Show that the modified Euler's method is convergent.

3. Starting with the equation $y' = \lambda y$, show that the modified Euler's method is relatively stable.

4. Apply the fourth order Runge-Kutta method to the equation $dy/dx = \mu y$, $y(x_0) = y_0$ and show that the range of absolute stability is

$$-2.78 < \mu h < 0.$$

5. Find the range of absolute stability of the equation

$y' + 10y = 0$, $y(0) = 1$, using

(a) Euler's method, (b) Runge-Kutta method.

6. Show that the local truncation errors in the Milne's predictor and corrector formulae are

$\frac{14}{45} h^5 y^u$ and $-\frac{1}{90} h^5 y^u$, respectively.

10.16 Boundary Value Problems

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types:

(i) $\frac{d^2y}{dx^2} + \lambda(x)\frac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a$,

$y(x_n) = b.$

(*ii*) $\frac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b.$

There exist two numerical methods for solving such boundary value problems. The first one is known as the *finite difference method* which makes use of finite difference equivalents of derivatives. The second one is called the *shooting method* which makes use of the techniques for solving initial value problems.

10.17 Finite-Difference Method

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The *finite-difference approximations to the various derivatives are derived as under:*

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots \quad (1)$$

and $$y(x-h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots \quad (2)$$

Equation (1) gives

$$y'(x) = \frac{1}{h}\left[y(x+h) - y(x)\right] - \frac{h}{2}y''(x) - \cdots$$

i.e., $$y'(x) = \frac{1}{h}\left[y(x+h) - y(x)\right] + O(h)$$

which is the *forward difference approximation of* $y'(x)$ with an error of the order h.

Similarly (2) gives

$$y'(x) = \frac{1}{h}\left[y(x) - y(x-h)\right] + O(h)$$

which is the *backward difference approximation of* $y'(x)$ with an error of the order h.

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h}\left[y(x+h) - y(x-h)\right] + O(h^2)$$

which is the *central-difference approximation of* $y'(x)$ with an error of the order h^2. Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2}\left[y(x+h) - 2y(x) + y(x-h)\right] + O(h^2)$$

which is the *central difference approximation of* $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence *the working expressions for the central difference approximations to the first four derivatives of* y_i *are as under*:

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) \tag{3}$$

$$y_i'' = \frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) \tag{4}$$

$$y_i''' = \frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right) \tag{5}$$

$$y_i^{iv} = \frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right) \tag{6}$$

NOTE **Obs.** *The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.*

EXAMPLE 10.34

Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$.

Solution:

We divide the interval (0, 1) into four sub-intervals so that $h = 1/4$ and the pivot points are at $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$.

Then the differential equation is approximated as

$$\frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] = x_i + y_i$$

or $\quad 16y_{i+1} - 33y_i + 16_{i-1} = x_i,\ i = 1, 2, 3.$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}; 16y_3 - 33y_2 + 16y_1 = \frac{1}{2}; - 33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$y_1 = -0.03488,\ y_2 = -0.05632,\ y_3 = -0.05003.$

NOTE

Obs. *The exact solution being* $y(x) = \frac{\sinh x}{\sinh 1} - x$, *the error at each nodal point is given in the table below:*

x	*Computed value* $y(x)$	*Exact value* $y(x)$	*Error*
0.25	– 0.03488	– 0.03505	0.00017
0.5	– 0.05632	– 0.05659	0.00027
0.75	– 0.05003	– 0.05028	0.00025

EXAMPLE 10.35

Using the finite difference method, find $y(0.25)$, $y(0.5)$, and $y(0.75)$ satisfying the differential equation $\frac{d^2y}{dx^2} + y = x$, subject to the boundary conditions $y(0) = 0$, $y(1) = 2$.

Solution:

Dividing the interval (0, 1) into four sub-intervals so that $h = 0.25$ and the pivot points are at $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, and $x_4 = 1$.

The given equation $y''(x) + y(x) = x$, is approximated as

$$\frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] + y_i = x_i$$

or $$16y_{i+1} - 31y_i + 16y_{i-1} = x_i \qquad (i)$$

Using $y_0 = 0$ and $y_4 = 2$, (i) gives the system of equation,

$$(i = 1)\ 16y_2 - 31y_1 = 0.25; \qquad (ii)$$

$$(i = 2)\ 16y_3 - 31y_2 + 16y_1 = 0.5 \qquad (iii)$$

$$(i = 3)\ 32 - 31y_3 + 16y_2 = 0.75,\ i.e.,\ -31y_3 + 16y_2 = -31.25 \qquad (iv)$$

Solving the equations (*ii*), (*iii*), and (*iv*), we get

$$y_1 = 0.5443,\ y_2 = 1.0701,\ y_3 = 1.5604$$

Hence $y\ (0.25) = 0.5443$, $y(0.5) = 1.0701$, $y(0.75) = 1.5604$

EXAMPLE 10.36

Determine values of y at the pivotal points of the interval (0, 1) if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).

Solution:

Here $h = 1/3$ and the pivotal points are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$. The corresponding y-values are $y_0(= 0)$, y_1, y_2, $y_3(= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right) + 81y_i = 81x_i^2$$

or $\quad y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2,\ i = 1, 2$

At $\quad i = 1,\ y_3 - 4y_2 + 7y_1 - 4y_0 + y{-1} = 1/9$

At $\quad i = 2,\ y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $-4y_2 + 7y_1 + y{-1} = 1/9$ (*i*)

$$y_4 + 7y_2 - 4y_1 = 4/9 \qquad (ii)$$

Regarding the conditions $y_0'' = y_3'' = 0$, we know that

$$yi'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$, $\quad y_0'' = 9\ (y_1 - 2y_0 + y{-1})$ or $y{-1} = -y_1$ $[\because y_0 = y_0'' = 0]$ $\quad$ (*iii*)

At $i = 3$, $\quad y_3'' = 9(y_4 - 2y_3 + y_2)$ or $y_4 = -y_2$ $[\because y_3 = y_3'' = 0]$ $\quad$ (*iv*)

Using (*iii*), the equation (*i*) becomes

$$-4y_2 + 6y_1 = 1/9 \qquad (v)$$

Using (*iv*), the equation (*ii*) reduces to

$$6y_2 - 4y_1 = 4/9 \qquad (vi)$$

Solving (*v*) and (*vi*), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

EXAMPLE 10.37

The deflection of a beam is governed by the equation $\dfrac{d^4y}{dx^4}+81y=\phi(x)$, where $f(x)$ is given by the table

x	1/3	2/3	1
$\phi(x)$	81	162	243

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Solution:

Here $h = 1/3$ and the pivotal points are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x3 = 1$. The corresponding y-values are $y_0(= 0)$, y_1, y_2, y_3.

The given differential equation is approximated to

$$\frac{1}{h^4}\left(y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}\right)+81y_i=\phi\left(x_i\right)$$

At $i=1$, $y_3-4y_2+7y_1-4y_0+y{-}1=1$ *(i)*

At $i=2$, $\quad y_4-4y_3+7y_2-4y_1+y_0=2$ *(ii)*

At $i=3$, $\quad y_5-4y_4+7y_3-4y_2+y_1=3$ *(iii)*

We have $\qquad y_0=0$ *(iv)*

Since $\qquad yi'=\dfrac{1}{2h}\left(y_{i+1}-y_{i-1}\right)$

$\therefore$ for $i=0$, $\qquad 0 \;=y_0'=\dfrac{1}{2h}\left(y_1-y-1\right)$ i.e., $y-1=y_1$ *(v)*

Since $\qquad yi''=\dfrac{1}{h^2}\left(y_{i+1}-\ 2y_i+y_{i-1}\right)$

$\therefore$ for $i=3$, $\qquad 0=y_3''=\dfrac{1}{h^2}\left(y_4-2y_3+y_2\right)$, i.e., $y_4=2y_3-y_2$ *(vi)*

Also $\qquad y_i'''=\dfrac{1}{2h^3}\left(y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2}\right)$

$\therefore$ for $i=3$, $\qquad 0 \;=y_3'''=\dfrac{1}{2h^3}\left(y_5-2y_4+2y_2-y_1\right)$

i.e., $\qquad y_5=2y_4-2y_2+y_1$ *(vii)*

Using *(iv)* and *(v)*, the equation *(i)* reduces to

$$y_3-4y_2+8y_1=1 \qquad (viii)$$

Using (*iv*) and (*vi*), the equation (*ii*) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \qquad (ix)$$

Using (*vi*) and (*vii*), the equation (*iii*) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \qquad (x)$$

Solving (*viii*), (*ix*), and (*x*), we get

$$y_1 = 8/13,\ y_2 = 22/13,\ y_3 = 37/13.$$

Hence $y(1/3) = 0.6154$, $y(2/3) = 1.6923$, $y(1) = 2.8462$.

10.18 Shooting Method

In this method, the given boundary value problem is first transformed to an initial value problem. Then this initial value problem is solved by Taylor's series method or Runge-Kutta method, etc. Finally the given boundary value problem is solved. The approach in this method is quite simple.

Consider the boundary value problem

$$y''(x) = y(x),\ y(x) = A,\ y(b) = B \qquad (1)$$

One condition is $y(a) = A$ and let us assume that $y'(a) = m$ which represents the slope. We start with two initial guesses for m, then find the corresponding value of $y(b)$ using any initial value method.

Let the two guesses be m_0, m_1 so that the corresponding values of $y(b)$ are $y(m_0, b)$ and $y(m_1, b)$. Assuming that the values of m and $y(b)$ are linearly related, we obtain a better approximation m_2 for m from the relation:

$$\frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

This gives

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m \quad b) \quad y(b)}{y(m_1, b) \quad y(m_0, b)} \qquad (2)$$

We now solve the initial value problem

$$y''(x) = y(x),\ y(a) = A,\ y'(a) = m_2$$

and obtain the solution $y(m_2, b)$.

To obtain a better approximation m_3 for m, we again use the linear relation (2) with $[m_1, y(m_1, b)]$ and $[m_2, y(m_2, b)]$. This process is repeated until the value of $y(m_i, b)$ agrees with $y(b)$ to desired accuracy.

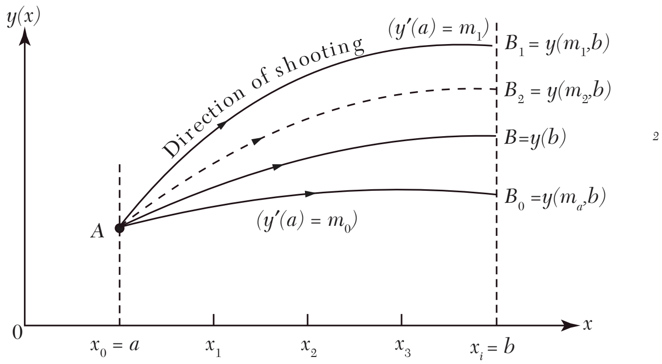

FIGURE 10.5

NOTE **Obs.** *This method resembles an artillery problem and as such is called the shooting method (Figure 10.5). The speed of convergence in this method depends on our initial choice of two guesses for m. However, the shooting method is quite slow in practice. Also this method is quite tedius to apply to higher order boundary value problems.*

EXAMPLE 10.38

Using the shooting method, solve the boundary value problem:

$$y''(x) = y(x),\ y(0) = 0 \text{ and } y(1) = 1.17.$$

Solution:

Let the initial guesses for $y'(0) = m$ be $m_0 = 0.8$ and $m_1 = 0.9$. Then $y''(x) = y(x)$, $y(0) = 0$ gives

$$y'(0) = m \qquad y''(0) = y(0) = 0$$

$$y'''(0) = y'(0) = m, \qquad y^{iv}(0) = y''(0) = 0$$

$$yv(0) = y'''(0) = m, \qquad yvi(0) = yiv(0) = 0$$

and so on.

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)$$

$\therefore \qquad y(1) = m(1 + 0.1667 + 0.0083 + 0.0002 + \cdots) = m\,(1.175)$

For $m_0 = 0.8$, $y(m_0, 1) = 0.8 \times 1.175 = 0.94$

For $m_1 = 0.9$, $y(m_1, 1) = 0.9 \times 1.175 = 1.057$

Hence a better approximation for m, i.e., m_2 is given by

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1,1) - y(1)}{y(m_1,1) - y(m_0,1)}$$

$$= 0.9 - (0.1)\frac{1.057 - 1.175}{1.057 - 0.94} = 0.9 + 0.10085 = 1.00085$$

which is closer to the exact value of $y'(0) = 0.996$

We now solve the initial value problem

$$y''(x) = y(x),\ y(0) = 0,\ y'(0) = m_2.$$

Taylor's series solution is given by

$$y(m_2, 1) = m_2\,(1.175) = 1.1759$$

Hence the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.17$.

Exercises 10.8

1. Solve the boundary value problem for $x = 0.5$:

$$\frac{d^2y}{dx^2} + y + 1 = 0, y(0) = y(1) = 0. \qquad \text{(Take } n = 4\text{)}$$

2. Find an approximate solution of the boundary value problem:

$y'' + 8(\sin^2 \pi y)\, y = 0,\ 0 \le x \le 1,\ y(0) = y(1) = 1.$ (Take $n = 4$)

3. Solve the boundary value problem:

$xy'' + y = 0,\ y(1) = 1,\ y(2) = 2.$ (Take $n = 4$)

4. Solve the equation $y'' - 4y' + 4y = e^{3x}$, with the conditions $y(0) = 0$, $y(1) = -2$, taking $n = 4$.

5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.

6. Solve the boundary value problem

$$y'' + xy' + y = 3x^2 + 2,\ y(0) = 0,\ y(1) = 1.$$

7. The boundary value problem governing the deflection of a beam of length three meters is given by

$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4, y(0) = y'(0) = y(3) = y'(3) = 0.$$

The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$).
Determine y at the pivotal points $x = 1$ and $x = 2$.

8. Solve the boundary value problem,

$$\frac{d^4y}{dx^4} + 81y = 729x^2 y(0) = y'(0) = y''(1) = y'''(1) = 0. \text{ Use } n = 3$$

9. Solve the equation $y^{iv} - y''' + y = x^2$, subject to the boundary conditions $y(0) = y'(0) = 0$ and $y(1) = 2,\ y'(1) = 0.$ (Take $n = 5$).

10. Apply shooting method to solve the boundary value problem

$$\frac{d^2y}{dx2} = y,\ y(0) = 0 \text{ and } \mathrm{y}(1) = 1.1752.$$

11. Using shooting method, solve the boundary value problem

$$\frac{d^2y}{dx2} = 6y^2,\ y(0) = 1,\ y(0.5) = 0.44$$

10.19 Objective Type of Questions

Exercises 10.9

Select the correct answer or fill up the blanks in the following questions:

1. Which of the following is a step by step method:

(a) Taylor's (b) Adams-Bashforth
(c) Picard's (d) None.

2. The finite difference scheme for the equation $2y'' + y = 5$ is ....

3. If $y' = x + y$, $y(0) = 1$ and $y^{(1)} = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is ......

4. The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is ....

5. Taylor's series for solution of first order ordinary differential equations is ....

6. The disadvantage of Picard's method is ....

7. Given y_0, y_1, y_2, y_3, Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is ....

8. The second order Runge-Kutta formula is ....

9. Adams-Bashforth predictor formula to solve $y' = f(x, y)$, given $y_0 = y(x_0)$ is .... .

10. The Runge-Kutta method is better than Taylor's series method because ....

11. To predict Adam's method atleast ...... values of y, prior to the desired value, are required.

12. Taylor's series solution of $y' - xy = 0$, $y(0) = 1$ upto x^4 is ....

13. If dy/dx is a function of x alone, the fourth order Runge-Kutta method reduces to ....

14. Milne's Predictor formula is ....

15. Adam's Corrector formula is ....

16. Using Euler's method, $dy/dx = (y - 2x)/y$, $y(0) = 1$; gives $y\ (0.1) =$ ..... .

17. $\dfrac{d^2y}{dx^2} + y^2\dfrac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations ...... and ....

18. The formula for the fourth order Runge-Kutta method is ....

19. Taylor's series method will be useful to give some ...... of Milne's method.

20. The names of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are ....

21. In the derivation of the fourth order Runge-Kutta formula, it is called fourth order because .....

22. If $y' = x - y$, $y(0) = 1$ then by Picard's method, the value of $y^{(1)}(1)$ is ....
(*a*) 0.915 (*b*) 0.905 (*c*) 1.091 (*d*) none.

23. The finite difference formulae for $y'(x)$ and $y''(x)$ are ....

24. If $y' = -y$, $y(0) = 1$, then by Euler's method, the value of $y(1)$ is
(*a*) 0.99 (*b*) 0.999 (*c*) 0.981 (*d*) none.

25. Write down the difference between initial value problem and boundary value problem ..... .

26. Which of the following methods is the best for solving initial value problems:
(*a*) Taylor's series method
(*b*) Euler's method
(*c*) Runge-Kutta method of the fourth order
(*d*) Modified Euler's method.

27. The finite difference scheme of the differential equation $y'' + 2y = 0$ is .....

28. Using the modified Euler's method, the value of $y(0.1)$ for
$\frac{dy}{dx} = x - y, y(0) = 1$ is
(*a*) 0.809 (*b*) 0.909 (*c*) 0.0809 (*c*) none.

29. The multi-step methods available for solving ordinary differential equations are ....

30. Using the Runge Kutta method, the value of $y(0.1)$ for $y' = x - 2y$, $y(0) = 1$, taking $h = 0.1$, is ......
(*a*) 0.813 (*b*) 0.825 (*c*) 0.0825 (*c*) none.

31. In Euler's method, if h is small the method is too slow, if h is large, it gives inaccurate value. (True or False)

32. Runge-Kutta method is a self-starting method. (True or False)

33. Predictor-corrector methods are self-starting methods. (True or False)